# Rutgers University: Algebra Written Qualifying Exam

**January 2013: Day 1 Problem 3 Solution**

**Exercise.** Let $A$ and $B$ be complex square matrices such that $AB - BA = B$. Prove that $AB - BA$ is not invertible.

Solution.

Suppose for contradiction that $AB - BA$ is invertible.

Then $B$ is an invertible complex square matrix.

$$\begin{aligned} AB - BA = B \quad &\implies \quad AB = BA + B \\ &\phantom{\implies} \quad \phantom{AB} = B(A+I) \\ &\implies \quad A = B(A+I)B^{-1} \end{aligned}$$

Therefore, $A$ and $A + I$ are similar matrices

$\implies$ $A$ and $A+I$ have the same eigenvalues in $\mathbb{C}$ since similar matrices share the same eigenvalues

But this is impossible!

**ALTERNATIVELY:**

$$\begin{aligned} & & AB - BA &= B & \\ &\implies & A - BAB^{-1} &= I & \\ &\implies & tr(A - BAB^{-1}) &= tr(I) & \\ & & &= n & \text{if } A, B\ n \times n \\ &\text{But} & tr(A - BAB^{-1}) &= tr(A) - tr(BAB^{-1}) & \\ & & &= tr(A) - tr(BB^{-1}A) & \\ & & &= tr(A) - tr(A) & \\ & & &= 0 & \\ & & &\neq n, & \text{a contradiction!} \end{aligned}$$

Thus, $AB - BA$ is not invertible.